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LETTER TO THE EDITOR

New series of integrable vertex models through a unifying approach

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Abstract

Applying a unifying Lax operator approach to statistical systems, a new class of integrable vertex models based on quantum algebra is proposed, which exhibits a rich variety for generic q, q roots of unity and $q \to 1$. Exact solutions are formulated through an algebraic Bethe ansatz, and the novel possibility of hybrid vertex models is introduced.

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1. Introduction

D-dimensional quantum systems are known to be related to $(1 + D)$-dimensional classical statistical models. Naturally, this is also true for $D = 1$, where one finds an exclusive class of model, known as integrable systems, allowing exact solutions. A celebrated example of such a relation is that between the XYZ quantum spin-$\frac{1}{2}$ chain and the eight-vertex statistical model, and similarly between the XXZ chain and the six-vertex model [1]. The Hamiltonian H_s of the integrable quantum spin chain is given through its transfer matrix as $\ln \tau(u) = I + uH_s + O(u^2)$, while the partition function Z of the related vertex model is constructed from $\tau(u)$ as $Z = \mathrm{tr}(\tau(u)^M)$. Moreover, both these models share the same quantum R-matrix and have the same representation for the transfer matrix, commutativity, of which $[\tau(u), \tau(v)] = 0$ guarantees their integrability.

It is therefore rather surprising to note that, in spite of such a close connection between these two integrable systems, their starting formulation conventionally follows two different routes. Quantum systems are usually defined by their Lax operators $L_{al}(u)$, which satisfy the quantum Yang–Baxter equation (YBE) $R_{ab}(u - v)L_{al}(u)L_{bl}(v) = L_{bl}(v)L_{al}(u)R_{ab}(u - v)$, together with its associated R-matrix. A vertex model, on the other hand, is described by its Boltzmann weights given generally through the elements of the R-matrix alone, which solves the YBE $R_{ab}(u - v)R_{al}(u)R_{bl}(v) = R_{bl}(v)R_{al}(u)R_{ab}(u - v)$. However, such a difference in these approaches, the reason for which seems to be rather historical, puts certain restrictions on the two-dimensional vertex models by assuming that their vertical (v) and horizontal (h) links, which are related to the auxiliary and quantum spaces respectively, are equivalent. As a consequence, while a rich variety of integrable quantum systems with a wide range of interactions involving spin, fermionic, bosonic and canonical variables does exist, the integrable vertex models are confined mostly to those quantum models that exhibit regularity property expressed through the permutation operator, $L_{al}(0) = P_{al}$, and hence correspond to local Hamiltonians with nearest-neighbour (NN) interactions. Therefore, the well-known examples of the integrable vertex models, apart from those mentioned above, appear to be limited mainly to models such as the five-vertex model [2], six-vertex model in external fields [3], 19-vertex model connected with the Babujian–Takhtajan spin-1 chain [4] and the vertex models equivalent to the Hubbard model [5], supersymmetric t–J model [6], Bariev chain [7], etc, all exhibiting only NN interactions.

The basic idea of this letter, however, is to exploit fully the equivalence between statistical and quantum systems and to construct a new class of integrable vertex models by applying a unifying scheme designed originally for quantum models [8]. In the original scheme, an ancestor model was proposed for generating integrable quantum systems as its various descendant realizations. To describe our vertex models, we start in analogy also with the generalized Lax operator [8]

$$L(u) = \begin{pmatrix} c_1^+ q^{S^3+u} + c_1^- q^{-(S^3+u)} & 2\sin\alpha S^- \\ 2\sin\alpha S^+ & c_2^+ q^{-(S^3-u)} + c_2^- q^{S^3-u} \end{pmatrix} \qquad q = \mathrm{e}^{\mathrm{i}\alpha} \tag{1}$$

linked with the underlying quantum algebra

$$[S^3, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = (M^+[2S^3]_q + M^-[[2S^3]]_q) \qquad [M^\pm, \cdot] = 0. \tag{2}$$

Here $[x]_q \equiv \frac{\sin(\alpha x)}{\sin\alpha}$, $[[x]]_q \equiv \frac{\cos(\alpha x)}{\sin\alpha}$ and the central elements $M^\pm$ are related to the other set of such elements appearing in the L-operator as $M^\pm = \pm\frac{1}{2}\sqrt{\pm 1}(c_+^+ c_-^- \pm c_+^- c_-^+)$. It is important to notice that equation (2) is a q-deformed quadratic algebra, which generalizes both q-spin and q-boson algebras and, in fact, follows from the quantum YBE representing integrability condition. We define the Boltzmann weights (BWs) of our vertex models not by the R-matrix, as is conventionally done, but through the elements of the Lax operator, $L_{ab}^{j,k}(u) = \omega_{a,j;b,k}(u)$, by using matrix representations of the general algebra (2) in (1). These generalized BWs generate a unified vertex model, which through possible reductions yields a new series of vertex models linked with different underlying algebras, their representations and choices of the central elements. Prominent examples of such integrable statistical systems are a rich collection of vertex models linked to q-spin and q-boson with generic q, q roots of unity and $q \to 1$. In all these models, the h and v links, contrary to the usual approach, may become inequivalent and independent at every vertex point. Since we consider here 2×2 Lax operators, the h links admit only two values: $a, b \in [+, -]$. The v links, on the other hand, have richer possibilities with $j, k \in [1, D]$, depending on dimension D of the matrix representation of the q-algebras (see figure 1). The familiar ice-rule is generalized here as the 'colour' conservation $a + j = b + k$ for determining non-zero BWs. However, the crucial partition function of the models is given as usual by $Z = \sum_{config} \prod_{a,b,j,k} \omega_{a,j;b,k}(u)$.

An important point to note is that, unlike the traditional approach, the Lax operators related to such vertex models do not coincide with their R-matrix, do not comply with the regularity condition, and do not correspond in general to quantum Hamiltonians with NN interactions. Moreover, since our vertex models belonging to the same class have the same R-matrix, we can generate another rich series of integrable models, namely hybrid vertex models, by combining any number of them in a row (see figure 1).

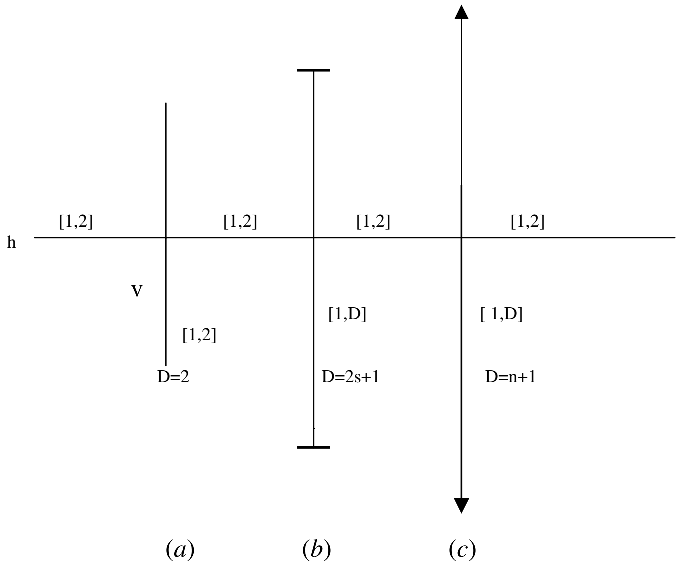

Figure 1. Integrable vertex models with horizontal (h) links taking two values, while the vertical (v) links may have D possible values: (a) six-vertex; (b) q-spin vertex; and (c) q-boson vertex models. Combining (a), (b) and (c), an integrable hybrid model may be formed. $q^p = \pm 1$ gives $D = p$ in (b) and (c).

The eigenvalue solution of the transfer matrix needed for constructing the partition function for all these vertex models can also be found exactly through the algebraic Bethe ansatz in a unifying way.

2. Unified vertex model

In accordance with our primary goal, we discover an explicit matrix representation for the basic operators $S^{\pm}, S^3$

$$\langle s, \bar{m}|S^3|m, s\rangle = m\delta_{m,\bar{m}} \qquad \langle s, \bar{m}|S^{\pm}|m, s\rangle = f_s^{\pm}(m)\delta_{m\pm 1,\bar{m}} \tag{3}$$

with $f_s^+(m) = f_s^-(m+1) = (\kappa + [s-m]_q(M^+[s+m+1]_q + M^-[[s+m+1]]_q))^{\frac{1}{2}}$ having additional parameters κ, s. It may be checked that equation (3) indeed gives an exact representation of the general q-deformed algebra (2) for arbitrary values of the central elements $M^{\pm}$. Therefore, the BW may be constructed from the matrix representation of the generalized Lax operator (1) by using equation (3) in the form

$$\omega_{\pm,j;\pm,j}(u) = c_{\pm}^{+}\,\mathrm{e}^{\mathrm{i}\alpha(u\pm m)} + c_{\pm}^{-}\,\mathrm{e}^{-\mathrm{i}\alpha(u\pm m)} \qquad \omega_{+,j;-,j-1} = \omega_{-,j-1;+,j} = 2f_s^+(m)\sin\alpha \tag{4}$$

with $m = s + 1 - j$, $j = 1, 2, \ldots, D$. The BW parameterized as (4) would now generate a unified $(4D-2)$-vertex model, representing a new series with arbitrary parameters $c_{\pm}^{\pm}$, s and κ. These models, and naturally all others obtained below through various reductions, are integrable statistical models and share the same R-matrix, which is given by that of the well-known six-vertex model [1].

Note that, though in general the dimension D of the matrices (3) is infinite, it may become truncated through the possible appearance of zero-normed states. To analyse this important effect we observe that, since $[0]_q = 0$, one obtains $f_s^+(m=s) = 0$ for $\kappa = 0$, recovering the familiar 'vacuum' state: $S^+|s, s\rangle = 0$. However, due to the presence of the second term, here one obtains $f_s^-(m=-s) = ([2s+1]_q(M^+[0]_q + M^-[[0]]_q))^{\frac{1}{2}} \neq 0$, and unlike the spin representation we have $S^-|m,s\rangle \neq 0$ for all m. Therefore, this creates an infinite tower of states by the action of the lowering operator S^-, as typical for the bosonic representation. This also signals the fact that algebra (2) includes both q-spin and q-boson and therefore their representations must be derivable from equation (3) as particular cases.

3. q-spin vertex model

It is straightforward to check that for $M^+ = 1$, $M^- = 0$, our unifying algebra (2) reduces to the well-known $U_q(su(2))$ quantum spin algebra [9] and at the same time equation (3) reproduces the known q-spin representation. Therefore, the corresponding BW may be obtained from equation (4) for a consistent choice $c_+^\pm = c_-^\pm = \mp\mathrm{i}$, as

$$\omega_{\pm,j;\pm,j}(u) = [u \pm m]_q \qquad \omega_{+,j;-,j-1} = \omega_{-,j-1;+,j} = f_s^{+(qspin)}(m) \qquad m = s+1-j$$

with $f_s^{\pm(qspin)}(m) = ([s \mp m]_q[s \pm m + 1]_q)^{\frac{1}{2}}$. In this case, the truncation $S^\pm|m = \pm s, s\rangle = 0$ typical for spin models and hence the familiar $D = 2s+1$-dimensional representation naturally arise, which produces therefore a series of q-spin $(8s+2)$-vertex models. The six-vertex model is clearly recovered at $s = \frac{1}{2}$, while $s = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ yield new 10, 14, 18, 22, ...-vertex models (see figures 1(*a*) and (*b*)).

The quantum systems related to such statistical models may be represented in general by interacting q-spins with nonlocal interactions. In particular, since the well-known sine-Gordon model is a realization of the q-spin [10], the vertex models constructed with non-zero κ and having infinite D will be related to the quantum integrable lattice sine-Gordon model [11].

4. q-boson vertex model

We find that q-bosonic algebra [12] can also be derived as a subalgebra of equation (2) for the complementary choice $M^+ = \sin\alpha$, $M^- = \cos\alpha$ by denoting $S^+ = \rho A$, $S^- = \rho A^\dagger$, $S^3 = -N$, $\rho = (\cot\alpha)^{\frac{1}{2}}$. Therefore, the same choice derives the matrix representation for the q-boson directly from equation (3) with the assumption that $\kappa = s = 0$ and $n = -m$, yielding

$$f_0^{-(qbos)}(n) = ([1+n]_q[[-n-1]]_q)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}[1+n]_{q^2}^{\frac{1}{2}}$$
$$f_0^{+(qbos)}(n) = f_0^{-(qbos)}(n-1) = \frac{1}{\sqrt{2}}[n]_{q^2}^{\frac{1}{2}}.$$

Consequently for a consistent solution $c_\pm^+ = 1$, $c_\pm^- = \mp\mathrm{i}\mathrm{e}^{\pm\mathrm{i}\alpha}$ from equation (4) we can derive the BW as

$$\omega_{\pm,j;\pm,j}(u) = \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\alpha\phi}[u \mp (j+\phi-1)]_q, \qquad \phi = \frac{1}{2}\left(1+\frac{\pi}{2\alpha}\right)$$
$$\omega_{+,j;-,j-1} = \omega_{-,j-1;+,j} = f_0^{+(qbos)}(j-1) = \frac{1}{\sqrt{2}}[j-1]_{q^2}^{\frac{1}{2}}. \tag{5}$$

It is obvious that, apart from the vacuum state $|0\rangle$ with $f_0^{+(qbos)}(0) = \frac{1}{\sqrt{2}}[0]_{q^2}^{\frac{1}{2}} = 0$, we can have no other zero-normed states and the q-bosonic representation, like the standard boson, is semi-infinite with $D = n+1$. The integrable $(4n+2)$-vertex model linked to the q-boson (figure 1(*c*)) that we construct using equation (5) would therefore be related to the lattice version of the quantum derivative nonlinear Schrödinger model (DNLS), which exhibits a q-bosonic realization [13].

5. Vertex models with q roots of unity

An excellent possibility for regulating the dimension of the matrix representation opens up when $q = e^{i\alpha}$ is chosen as a solution of $q^p = \pm 1$ with the parameter α taking discrete values $\alpha_a = \pi \frac{a}{p}, a = 1, 2, \ldots, p - 1$ [14]. Note, however, that when some values of a become a factor of p one faces a situation with $q^{\frac{p}{a}} = \pm 1$. Therefore, to avoid such complications we suppose p to be prime in our present discussion. For further analysis, we focus on the action of S^- assuming $\kappa = 0$ in equation (3), $S^-|m = -\bar{s}, s\rangle = ([s+\bar{s}+1]_q(M^+[s-\bar{s}]_q + M^-[[s-\bar{s}]]_q))^{\frac{1}{2}}$, and we observe that, due to $[p]_q = \sin\alpha_a p = 0$, unlike generic q we can now obtain $S^-|-\bar{s}, s\rangle = 0$ at $\bar{s} = p - (s+1)$, which reduces matrix (3) to a finite-dimensional representation. Therefore, the BW obtained from equation (4) for this case would generate another series of unified K-vertex models having finite $\mathrm{K} = 4p - 2$ configurations at every vertex point. Moreover, since for a fixed p there can be $p - 1$ different α_a, each of these discrete values describes a different set of BWs and hence a novel model.

Consequently, at particular reductions as analysed above, we obtain the corresponding series of new vertex models linked with q-spin or q-boson, but now having finite configuration space determined by p. Thus, for the q-spin with fixed $p, 0 < p < 2s + 1$, in place of a $(8s+2)$-vertex model for generic q, one obtains a $p - 1$ number of different $(4p - 2)$-vertex models and the related representations including the case $p > 2s + 1$ become more involved [14]. The corresponding BWs defining these models should, however, be given by their same generic form, though using discrete α_a values. As, for example, in case of $s = \frac{5}{2}$ with $q^5 = -1$, instead of a 22-vertex model one obtains four different 18-vertex models for distinct values of $\alpha_a = \pi\frac{a}{5}, a = 1, 2, 3, 4$. Noticeably, as a quantum model, the q-spin with q roots of unity is realized as the restricted sine-Gordon model [15].

The situation becomes more interesting when applied to the q-boson with finite p, since now together with the standard vacuum we also obtain $A^\dagger\big|n = \frac{p}{2} - 1\big\rangle = \frac{1}{\sqrt{2}}\big[\frac{p}{2}\big]_{q^2}^{\frac{1}{2}}\big|\frac{p}{2}\big\rangle = 0$, yielding finite $\left(\frac{p}{2} \times \frac{p}{2}\right)$ matrix representations for the q-bosonic operators $A, A^\dagger$. As a result, we obtain an intriguing series of $(2p - 2)$-vertex models with BWs described by the same form (5) as for the generic q-bosonic case, but with different possible parameter values $q = e^{i\alpha_a}, a = 1, 2, \ldots, p - 1$. The quantum model corresponding to such q-boson vertex models can be realized as the restricted DNLS model, which supports finite quasi-particle bound states [16].

6. Rational class of vertex models

At $q \to 1$ $(\alpha \to 0)$, on the other hand, the associated R-matrix goes to its known rational limit and the underlying algebra becomes undeformed with $M^\pm \to m^\pm$, reducing at the same time the unified model to its rational form. Consequently, taking the limits carefully, we may construct in a similar way the corresponding set of vertex models belonging to the rational class. Not going into detail, we mention only that the BWs of these vertex models can be obtained from the limiting values of equation (4) yielding $f_s^+(m) \to ((s-m)(m^+(m+s+1)+m^-))^{\frac{1}{2}}$. It is easy to check that the BWs for the vertex models related to the undeformed spin as well as the standard boson correspond to the particular values of the central elements, $m^+ = 1, m^- = 0$ and $m^+ = 0, m^- = 1$, respectively. Remarkably, the spin vertex model constructed in this way coincides with the similar higher s model obtained earlier through the fusion method [4], whereas the bosonic-vertex model apparently is a new model, linked to a quantum integrable lattice NLS model [17].

7. Hybrid vertex models

In constructing our vertex models, we have flatly assumed that in any model the same BW must be defined at every vertex point. An immediate generalization is therefore possible by relaxing this condition and by considering that the central elements $c_{\pm}^{\pm}$, as well as the spin parameters s appearing in equation (4), are different at different sites. As we have already stressed, vertex models obtained as various reductions of the same integrable unified model belong to the same class sharing the same R-matrix. Thus, the q-spin and q-boson vertex models are members of the trigonometric class, while the normal spin and boson models belong to the rational class. Based on this fact, therefore, we can construct a rich collection of hybrid models by combining different vertex models of the same class and inserting their defining BW along the vertex points $l = 1, 2, \ldots, N$ in a row, in any but fixed manner. Due to the association with the same R-matrix, the integrability of such statistical models would be naturally preserved.

Thus, for example, an alternate insertion of ten-vertex and six-vertex models results in a hybrid model, which is related to the known quantum model [18] involving spin-1 and spin-$\frac{1}{2}$ operators with next-NN interactions. More exotic hybrid models can be formed by arranging the BW for the q-spin and q-boson vertex models, alternatively or in any other way at different vertex points (see figure 1). Similarly one can construct a spin-boson hybrid vertex model by combining individual vertex models, which would correspond to a quantum chain of interacting spins and bosons involving next-NN couplings.

8. Unified solution

The construction of the unified vertex model through the generalized Lax operator also suggests a scheme for exactly solving the eigenvalue problem for the transfer matrix. Since the partition functions in turn can be determined from the knowledge of these eigenvalues, all vertex models obtained as particular cases and linked to (un-)deformed spin or (un-)deformed boson can also be solved in a unified way. There is a well-formulated algebraic Bethe ansatz method for exactly solving the eigenvalue problem of the transfer matrix, $\tau(u) = \mathrm{tr}_h\big(\prod_l^N L_l(u)\big)$, when the associated Lax operator as well as the R-matrix are given [19]. Therefore, since we have defined the BW through matrix representations of the Lax operator, and the R-matrix in our case is given by that of the well-known six-vertex model, we can derive the exact eigenvalues for the transfer matrix of our models as

$$\Lambda(u) = \omega_{+,1;+,1}^N(u)\prod_k^n g(u_k - u) + \omega_{-,1;-,1}^N(u)\prod_k^n g(u - u_k) \qquad g(u) = \frac{[u+1]_q}{[u]_q}. \tag{6}$$

with all possible solutions of $\{u_k\}$ to be determined from the Bethe equations

$$\left(\frac{\omega_{+,1;+,1}(u_k)}{\omega_{-,1;-,1}(u_k)}\right)^N = \prod_{k\neq j}^n \frac{[u_k - u_k + 1]_q}{[u_k - u_k - 1]_q} \qquad k = 1, 2, \ldots, n. \tag{7}$$

By analysing the structure of these equations we conclude that the factors involving the BW in both of these come from the action of the Lax operator on the pseudo vacuum, which is chosen as the direct product of the highest weight states with $j = 1$, i.e. $|m = s\rangle$. The remaining factors, on the other hand, originate from the R-matrix elements, which arise during diagonalization of the transfer matrix due to the use of the quantum YBE. Therefore, it is crucial to note that the only part given by the BW is model-dependent and defined for the vertex models by the diagonal entries in equation (4) with $j = 1$, while the remaining parts contributed by the R-matrix are the same for all our models from the same class. Consequently, the exact solutions for all models constructed here can be found in a systematic way from equations (6) and (7) by using corresponding reductions of the unified model (4).

The total number of solutions $\{\Lambda_\gamma(u)\}$ for the eigenvalues (6) should be D^N, which coincides with the number of possible eigenstates and gives the dimension of the vector space on which the transfer matrix acts. The partition function of the vertex models may therefore be given at the thermodynamic limit by $Z = \lim_{M,N\to\infty} \mathrm{tr}_v(\tau^M(u)) = \lim_{M,N\to\infty} \sum_{\gamma=1}^{D^N} \Lambda_\gamma^M(u)$. At this important limit, the Bethe equations (7) turn into an integral equation $V(u) = 2\pi\rho(u) + \int \rho(v)K(v,u)\,\mathrm{d}v$, with the known kernel of the six-vertex model [20]. Interestingly, all information about a particular model is encoded in the driving term only, which is expressed through $\frac{\omega_{+,1;+,1}(u)}{\omega_{-,1;-,1}(u)} = r\,\mathrm{e}^{\mathrm{i}P(u)}$ as $V(u) = P'(u)$. Therefore, knowing the explicit form of the BM, one can derive easily the equations for individual models.

To extract the solutions of the hybrid vertex models, however, the BW-dependent parts in the above equations should be slightly modified by generalizing the factors inhomogeneously as $\prod_\beta \left(\omega^{(\beta)}_{\pm,1;\pm,1}(u)\right)^{N_\beta}$, where N_β is the number of vertices of type β appearing in a row with the constraint $N = \sum_\beta N_\beta$.

A detailed investigation of individual models and the identification of their most probable states are important problems to be pursued.

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